

# A note on the post-Newtonian limit of quasi-local energy expressions

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January 12, 2013

## Abstract

An ‘effective’ quasi-local energy expression, motivated by the (relativistically corrected) Newtonian theory, is introduced in exact GR as the volume integral of all the source terms in the field equation for the Newtonian potential in static spacetimes. In particular, we exhibit a new post-Newtonian correction in the source term in the field equation for the Newtonian gravitational potential. In asymptotically flat spacetimes this expression tends to the ADM energy at spatial infinity as a *monotonically decreasing* set function. We prove its positivity in spherically symmetric spacetimes under certain energy conditions, and that its vanishing characterizes flatness. We argue that any physically acceptable quasi-local energy expression should behave qualitatively like this ‘effective’ energy expression in this limit.

## 1 Introduction

In non-gravitational classical field theories on flat Minkowski space the energy-momentum distribution of the matter fields is described by the symmetric energy-momentum tensor  $T_{ab}$  satisfying the dominant energy condition. Then the quasi-local energy of the matter fields  $E[D, K^a]$  with respect to some constant future pointing timelike unit vector field  $K^a$  (i.e. a time translational Killing vector of the flat spacetime) is defined to be the integral  $\int_D K_a T^{ab} t_b d\Sigma$  on the compact domain  $D \subset \Sigma$  with boundary  $\mathcal{S} := \partial D$  in some spacelike hypersurface  $\Sigma$ . Here  $t^a$  is the future pointing unit normal to  $\Sigma$  and  $d\Sigma$  is the natural volume element. As a consequence of the dominant energy condition this is not only positive definite, but also is a monotonically *increasing* set function: if  $D_1 \subset D_2$  then  $E[D_1, K^a] \leq E[D_2, K^a]$ . This implies, in particular, that in asymptotically flat configurations (when  $\Sigma$  extends to spatial infinity and the total energy  $E[\Sigma, K^a]$  is finite)  $E[D, K^a]$  tends to the total energy *from below* as  $D$  is blown up to exhaust  $\Sigma$ .

Although in general relativity there is no well defined energy-momentum *density* of the gravitational ‘field’, in asymptotically flat configurations its total (ADM) energy could be defined, and one of the greatest successes of classical general relativity in the last third of the 20th century is certainly the proof by Schoen and Yau [1] that the total gravitational energy is strictly positive definite. The logic of one of its simplest proofs, due to Witten [2] (and simplified and corrected by Nester [3]), is that we can rewrite the total energy as an integral of some expression (the so-called Sparling form [4], see also [5]) on a spacelike hypersurface, and by Witten’s *gauge condition* the integrand could be ensured to be pointwise strictly positive definite. Thus the negative definite part of the Sparling form in the integrand is a pure gauge term.

These results may yield the view that if the gravitational mass could be defined at the quasi-local level, then it would have to be not only positive definite, but also that in asymptotically flat spacetimes it would have to tend to the ADM mass as an *increasing* set function. (Note that while we can compare the quasi-local *masses*, i.e. scalars, we can compare the quasi-local *energies*, i.e. components of four-vectors on *different 2-surfaces* only in the presence of some extra structure, e.g. for spherically symmetric or stationary systems.) In fact, several specific quasi-local mass expressions exist which satisfy these requirements (viz. the Bartnik [6] mass, the mass built from the Dougan–Mason [7] energy-momenta, and the Misner–Sharp energy [8] for spherically symmetric configurations), or at least the second of these in certain special spacetime configurations (the Hawking [9] or the Geroch energies [10] and the Penrose mass [11]). However, there are other constructions (e.g. the Brown–York expressions [12], the Epp [13], the Kijowski–Liu–Yau [14, 15] and the Wang–Yau energies [16]) which tend to the ADM energy as *decreasing* set functions. (For a review and a more detailed discussion of the various quasi-local energy constructions and the extended literature, see e.g. [17].) These different monotonicity properties of the quasi-local mass/energy expressions generated some debate in the relativity community: whether a physically reasonable quasi-local energy expression should be monotonically *increasing* or *decreasing* near spatial infinity as it tends to the ADM energy. In fact, near spatial infinity the matter and the radiation can be neglected compared to the *negative definite* Newtonian gravitational binding energy. Therefore, increasing the domain of integration we have more and more *negative definite* contribution to the total energy, which therefore must be a decreasing set function (see also [18]).

Since near spatial infinity the dynamics of the fields and the gravitation dies off rapidly (typically as  $1/r^2$ ), in the analysis of the asymptotic behaviour of any quasi-local energy/mass expression it is natural to consider the spacetime to be *static* in the first approximation. The advantage of the existence of a static Killing vector is that it provides a geometrically preferred notion of time and a preferred foliation of spacetime by a geometrically distinguished family of extrinsically flat spacelike hypersurfaces. In fact, the Newtonian limit of general relativity is defined in this way in [19]. Identifying the Newtonian potential  $\phi$  with the logarithm of the length of the Killing field, we show that the exact field equations for this  $\phi$  take the form of a Poisson equation in which the source term contains not only the familiar rest mass density of the matter fields (the Newtonian source term), but (among others) minus the *square* of the gradient of  $\phi$  (post-Newtonian corrections) as well. Indeed, while the former is independent of  $c^{-2}$ , the latter is proportional to  $c^{-2}$  and can naturally be interpreted as the energy-density (and/or the trace of the spatial stress tensor) of the gravitational field itself. Thus, to avoid confusion, we use the phrase ‘post-Newtonian’ in the following sense: (1) the spacetime

is static and asymptotically flat, in which (2) when quantities are expanded as a series of  $c^{-2}$  then the zeroth order term are called Newtonian and the  $c^{-2k}$  order ones the  $k$ th order post-Newtonian corrections.

In the relativistically corrected Newtonian theory of gravity the volume integral of the source terms for the Newtonian potential gives a *well defined* (i.e. free of the ambiguities coming from the Galileo–Eötvös experiment) expression for the energy of the source plus gravity system even at a quasi-local level. Motivated by this observation, we show that in static spacetimes in *exact general relativity* the extra structures above are enough to introduce a notion of ‘effective’ quasi-local *energy* expression simply as the integral of the sum of all the source terms for  $\phi$ . This energy is shown to tend at spatial infinity to the ADM energy as a monotonically *decreasing* set function. We give an explicit form of this ‘effective’ quasi-local energy in static, spherically symmetric spacetimes, and under certain energy conditions we prove its positivity and its rigidity, i.e. that its vanishing implies flatness.

Since the idea behind the ‘effective’ quasi-local energy expression is exactly analogous to the quasi-local energy in the relativistically corrected Newtonian theory, we believe that the qualitative behaviour of the ‘effective’ quasi-local energy expression reflects some universality: we expect that any physically acceptable quasi-local *energy* expression in a static, asymptotically flat spacetime should tend to the ADM energy as a *decreasing* set function at spatial infinity.

In Section 2 we discuss the issue of energy both in Newtonian theory and in the relativistically corrected Newtonian theory. Then, in subsection 3.1, the Newtonian limit of Einstein theory is reviewed. We find that as a relativistic correction not only the energy density, but *the spatial stress of the gravitational field* also contributes to the effective source in the *exact* field equations for the Newtonian potential. We emphasize that this result is exact, i.e. no approximation is used. As far as we know this is a new post-Newtonian correction from GR, which has not been considered so far. Then the ‘effective’ quasi-local energy is introduced and analyzed in subsections 3.2 and 3.3, and compared with the Misner–Sharp and Brown–York expressions in subsection 3.4. Section 4 is devoted to the discussion of the potential implications for more general quasi-local energy-momentum expressions.

The sign conventions for the metric and the curvature of [20] are used. In particular, the signature of the spacetime metric is  $(+, -, -, -)$ , the curvature and Ricci tensors and the curvature scalar are defined by  $-R^a_{bcd}X^b v^c w^d := v^c \nabla_c(w^d \nabla_d X^a) - w^c \nabla_c(v^d \nabla_d X^a) - [v, w]^c \nabla_c X^a$ ,  $R_{bd} := R^a_{bad}$  and  $R := R_{ab}g^{ab}$ , respectively. Thus Einstein’s equations take the form  $G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} = -\kappa T_{ab} - \lambda g_{ab}$ , where  $\lambda$  is the cosmological constant and  $\kappa := 8\pi G/c^4$  with Newton’s gravitational constant  $G$  and the speed of light  $c$ ; i.e. we use the traditional units.

## 2 Gravitational energy in Newton’s theory

### 2.1 Newton’s theory

In a given inertial frame of reference the gravitational field is described by a scalar function  $\phi$  of the flat 3-space  $\mathbb{R}^3$ , for which the field equation is the Poisson equation

$$-D_a D^a \phi = 4\pi G \rho.$$

Here  $\rho : \mathbb{R}^3 \rightarrow [0, \infty)$  is the *rest mass* density of the matter (source), and  $D_a$  is the *flat* covariant derivative operator in the 3-space. Note that we use the negative definite flat metric  $h_{ab}$  here, consistent with our conventions. If  $D \subset \mathbb{R}^3$  is any open subset with compact closure and a smooth boundary  $\mathcal{S} = \partial D$ , then by the Gauss theorem and the field equation

$$m_D := \int_D \rho \, d^3x = \frac{1}{4\pi G} \oint_{\mathcal{S}} v^e (D_e \phi) \, d\mathcal{S}, \quad (2.1)$$

where  $v^e$  denotes the outward pointing unit normal of  $\mathcal{S}$  and  $d\mathcal{S}$  is the induced area element. Thus the *rest mass of the source* in Newtonian theory of gravity can be rewritten into a 2-surface integral, like the charge in electrostatics. Following the analogy with electrostatics, we can introduce the *energy density* and the *spatial stress* of the gravitational field itself, respectively, by

$$U := \frac{1}{8\pi G} h^{ab} (D_a \phi) (D_b \phi) = -\frac{1}{8\pi G} |D_a \phi|^2, \quad (2.2)$$

$$\Sigma_{ab} := \frac{1}{4\pi G} \left( (D_a \phi) (D_b \phi) - \frac{1}{2} h_{ab} (D_c \phi) (D^c \phi) \right). \quad (2.3)$$

In fact, the integral of  $U$  on  $\Sigma$  is just the work that we should do to form e.g. a spherical body by bringing particles together from infinity. Moreover, since gravitation is always attractive, in Newton's theory this is always a *binding* energy, and hence its sign is negative. The divergence of the stress tensor, together with the Poisson equation, yields the 'force density':  $D_a \Sigma^{ab} = \frac{1}{4\pi G} (D_a D^a \phi) D^b \phi = -\rho D^b \phi$ . Note also that the average 'gravitational pressure' is just one-third of the gravitational energy density:  $3P := -h^{ab} \Sigma_{ab} = U$ .

However, by the Galileo–Eötvös experiment there is an important difference between electrostatics and gravitation. Namely, by this experiment the inertial and gravitational masses of the particles are strictly proportional to each other. Hence, in particular, the gravitational and inertial masses of the test particles drop out from the equations of motion in the gravitational field, yielding an ambiguity even in the notion of the gravitational force  $D_e \phi$ : it is not possible, even in principle, to make a distinction between a *uniform* gravitational field and a *uniform* acceleration of the frame of reference. Therefore, at any *given* point of the 3-space the gravitational force  $D_e \phi$  can be transformed to any given value, e.g. to zero, by an appropriate change of the reference frame. Thus the ambiguity in the gravitational force is  $D_e \phi \mapsto D_e \phi + a_e$ , where  $a_e$  is an arbitrary *constant* covector field in 3-space. It is only the *second* derivative  $D_a D_b \phi$ , the tidal force, that has direct physical meaning. Consequently, the gravitational energy *density* (2.2) can also be transformed to any given non-positive value, e.g. to zero at any given point by an appropriate change of the frame of reference. Similarly, the spatial stress (2.3) is also vanishing at that given point. On the other hand, the gravitational energy density (as well as the spatial stress) can be transformed to zero on an extended, *open* subset of the 3-space *only if* the gravitational field is uniform there. If, however, we have some extra information about the structure of the gravitational field, e.g. that it is the gravitational field of a localized source, then the ambiguity can be removed from the gravitational force  $D_a \phi$  by requiring its vanishing at infinity.

To summarize, we see that even in Newtonian theory of gravity the gravitational energy and spatial stress *cannot be localized to a point*, just as a consequence of the Galileo–Eötvös experiment, and the *rest mass of the source* can also be written as a *2-surface integral*.

## 2.2 Two relativistic corrections to the source

According to the special theory of relativity mass and energy are not independent concepts, and we should associate a mass distribution to any distribution of energy in 3-space. In particular, in addition to the mass distribution  $u/c^2$  of the *internal energy* density  $u$  of the matter field, we should associate a mass distribution  $U/c^2$  to the energy density (2.2) of the gravitational field, too. However, according to the principle of equivalence, a consequence of the Galileo–Eötvös experiment, *any* mass distribution is a source of gravity, independently of the nature of the mass. Thus, in particular, both the internal energy density of the matter and the gravitational energy density contribute to the source of gravity. Therefore, the source term on the right hand side of the Poisson equation should be corrected, and the field equation for the relativistically corrected Newtonian theory of gravity could naturally be expected to be

$$-D_a D^a \phi = 4\pi G \left( \rho + \frac{1}{c^2} (u + U) \right). \quad (2.4)$$

Note that the gravitational energy density *reduces* the magnitude of the source, because that is a binding type energy. As a consequence of (2.4) we have that

$$E_D := \int_D (\rho c^2 + u + U) d^3x = \frac{c^2}{4\pi G} \oint_{\mathcal{S}} v^e (D_e \phi) d\mathcal{S}; \quad (2.5)$$

i.e. now it is the *total energy of the source plus gravity system* in a given domain  $D$  that can be rewritten into the form of a 2-surface integral. Note that while the gravitational energy *density* is ambiguous, this *quasi-local* expression for the energy of the matter plus gravity system is *free of this ambiguity*, just because the flux integral on  $\mathcal{S}$  of any constant covector field  $a_e$  is zero. This in itself already justifies the introduction and use of the quasi-local concept of energy in the study of gravitating systems. Moreover, in the source-free region (i.e. where  $\rho$  and  $u$  are vanishing)  $E_D$  is a *decreasing* set function, because then the integrand in the middle term is negative definite there. In particular, for a 2-sphere with radius  $r$  surrounding a localized spherically symmetric homogeneous source, the quasi-local energy is  $E_{D_r} = \frac{8\pi}{\kappa} \mathfrak{m} (1 + \frac{1}{2} \frac{\mathfrak{m}}{r} + \frac{\mathfrak{m}^2}{r^2}) + O(r^{-4})$ , where, in terms of the rest mass  $M$  and the radius  $R$  of the source, the *total mass parameter* is  $\mathfrak{m} = \frac{GM}{c^2} (1 - \frac{3}{5} \frac{GM}{c^2 R} + \frac{51}{140} (\frac{GM}{c^2 R})^2) + O(c^{-8})$ .

## 3 Gravitational energy in static spacetimes

### 3.1 The Newtonian limit of Einstein's theory and two more relativistic corrections

In [19], pp. 71–74, the Newtonian limit of Einstein's theory is defined through asymptotically flat, static configurations. Thus let the spacetime be static, and  $K^a$  be the (e.g. future pointing) timelike Killing field being orthogonal to the spacelike level sets  $t = \text{const}$  of a function  $t : M \rightarrow \mathbb{R}$ . These sets will be denoted by  $\Sigma_t$ , and we write  $K_a = g \nabla_a t$  for some function  $g$  on  $M$ . If  $f^2 := K_a K^a$ , then  $t^a := f^{-1} K^a$  is the future pointing unit timelike normal to the hypersurfaces  $\Sigma_t$ , and  $h_{ab} := g_{ab} - t_a t_b$  is the induced (negative definite) metric on  $\Sigma_t$ . Let  $D_a$  denote the corresponding intrinsic Levi-Civita covariant derivative operator on  $\Sigma_t$ . Then by a straightforward calculation it is shown in [19], pp 72, that the length  $f$  of the Killing field satisfies the 'field equation'  $h^{ab} D_a D_b f = f R_{ab} t^a t^b$ .

Next, let us define the energy density  $\mu := T_{ab}t^at^b$  of the matter fields seen by the static observers, decompose it into the sum of the rest mass and internal energy densities as  $\mu = c^2\rho + u$ , and write the trace of the energy-momentum tensor as  $T_{ab}g^{ab} = \mu - 3p$ . (Thus  $-3p$  denotes the trace of the spatial stress tensor  $\sigma_{ab} := P_a^c P_b^d T_{cd}$  with respect to the *negative* definite  $h_{ab}$ . Here,  $P_b^a$  denotes the obvious projection to  $\Sigma_t$ .) Introducing the scalar field  $\phi := c^2 \ln f$ , the field equation for  $f$  above, together with Einstein's equations, gives

$$-h^{ab}D_a D_b \phi = 4\pi G\rho + \frac{4\pi G}{c^2}\left(u + 3p - \frac{c^4\lambda}{4\pi G}\right) + \frac{1}{c^2}h^{ab}(D_a \phi)(D_b \phi). \quad (3.1)$$

Comparing this equation with (2.4) we see that apparently we recovered (2.4) with the relativistic correction term  $U/c^2$  dictated by the Galileo–Eötvös experiment (even with the correct sign), together with an additional relativistic correction: the trace of the spatial stress (as well as the cosmological constant) also contributes to the effective source (as already noted in [19]). Nevertheless, the relative weight of the gravitational energy density term in the effective source is *twice* that of (2.4): the last term on the right hand side of (3.1) is  $\frac{8\pi G}{c^2}U$  rather than the expected  $\frac{4\pi G}{c^2}U$ . However, as we saw in subsection 2.1, we can associate with the Newtonian gravitational field not only energy density but also *spatial stress*, and the corresponding average pressure  $P$  is one-third of the energy density. Thus the ‘extra’ gravitational energy density in (3.1) can be written as  $3P$ , i.e. the last term of (3.1) has the form  $\frac{4\pi G}{c^2}(U + 3P)$ . Hence there is a *fourth relativistic correction* to (2.4): the gravitational stress also contributes to the effective source of gravity. Note that this correction is obtained in the *exact* theory, independently of any approximation method. (In addition, the intrinsic geometry  $(\Sigma_t, h_{ab})$ , and hence the Laplacian  $-h^{ab}D_a D_b$ , is *not* flat. This can also be considered as an additional correction to (2.4), but it does not seem to be possible to formulate its deviation from the flat-space Laplacian of  $\phi$  in a *gauge invariant* way.)

Another (and perhaps more direct) derivation of (3.1) could be based on the evolution parts  $P_a^c P_b^d (G_{cd} + \kappa T_{cd} + \lambda g_{cd}) = 0$  of Einstein's equations in the standard 3+1 decomposition. If  $\xi^a = Nt^a + N^a$  is an evolution vector field which is compatible with the foliation  $\Sigma_t$ , then, using the Hamiltonian constraint  $t^a t^b (G_{ab} + \lambda g_{ab} + \kappa T_{ab}) = 0$ , the evolution equations are equivalent to

$$\begin{aligned} \dot{\chi}_{ab} = & N\left(-{}^3R_{ab} + 2\chi_{ac}\chi^c_b - \chi\chi_{ab}\right) + L_{\mathbf{N}}\chi_{ab} - D_a D_b N + \\ & + \lambda N h_{ab} + \kappa N\left(-\sigma_{ab} + \frac{1}{2}\sigma^c_c h_{ab} + \frac{1}{2}\mu h_{ab}\right), \end{aligned} \quad (3.2)$$

where  ${}^3R_{ab}$  is the Ricci tensor of the intrinsic 3-metric  $h_{ab}$  and  $L_{\mathbf{N}}\chi_{ab}$  is the Lie derivative of the extrinsic curvature  $\chi_{ab}$  of  $\Sigma_t$  along the shift vector field. Choosing the leaves  $\Sigma_t$  of the foliation to be the hypersurfaces to which the Killing field  $K^a$  is orthogonal the extrinsic curvature is vanishing, and choosing the evolution vector field to be the Killing field itself, the lapse will be the length of the Killing vector and the shift will be zero. Then (3.2) takes the form

$$-D_a D_b f = f\left({}^3G_{ab} + \kappa\sigma_{ab} + \frac{1}{2}\kappa(\mu + 3p)h_{ab}\right), \quad (3.3)$$

where  ${}^3G_{ab}$  denotes the Einstein tensor of the spatial metric  $h_{ab}$  and we used the Hamiltonian constraint

$$\frac{1}{2} {}^3R = \kappa\mu + \lambda. \quad (3.4)$$

(Since in static spacetimes the local momentum density  $t^a T_{ab} P_c^b$  of the matter fields is vanishing, the momentum constraint is satisfied identically.) Taking the trace of (3.3) and using  $\phi$  instead of  $f$  we recover (3.1). In what follows we need the full (3.3) rather than only its trace, and we consider (3.3)-(3.4) to be the field equations rather than only (3.1). In fact, in the static case (3.3)-(3.4) are equivalent to Einstein's equations, the field equations for the Newtonian potential  $\phi$  and the spatial metric  $h_{ab}$ .

### 3.2 The ‘effective’ quasi-local energy for static configurations and its spatial infinity limit

By the Galileo–Eötvös experiment *any* kind of energy is a source of gravitation. Thus, motivated by expression (2.5) of the relativistically corrected Newtonian theory, in exact general relativity in static spacetimes it is natural to *define* the total, ‘effective’ energy of the static matter+gravity system in a subset  $D \subset \Sigma_t$ , seen by the static observers  $t^a$ , as the integral of all the source terms on the right hand side of (3.1):

$$E_D := \int_D \left( \mu + 3p - \frac{c^4 \lambda}{4\pi G} - \frac{1}{4\pi G} |D_a \phi|^2 \right) d\Sigma = \frac{c^2}{4\pi G} \oint_{\mathcal{S}} v^a (D_a \phi) d\mathcal{S}. \quad (3.5)$$

Note that  $E_D$  contains not only the energy of the gravitational ‘field’ and (all kinds of) energy of the matter source, but the trace of the spatial stress of the source and the gravitational ‘field’ as well. Thus, apart from the cosmological term, the structure of the post-Newtonian part of the volume integral,  $u + U + 3(p + P)$ , shows some resemblance to enthalpy rather than to the internal energy density. However, this combination seems to deviate from the standard form of enthalpy, too, which would have the structure  $u + U + (p + P)$ . (For a more detailed discussion of the analogy of gravitational energy with the thermodynamical ones, see [14].) If the source is compactly supported in some  $D_0 \subset D \subset \Sigma_t$ , and the cosmological constant is non-negative, then outside  $D_0$  the total gravitational energy is strictly *decreasing* with increasing domain  $D$  of integration.

If the spacetime is asymptotically flat (in which case  $\lambda = 0$ ) and the hypersurfaces extend to the spatial infinity, then  $E_{D_R}$ , the quasi-local energy associated with the solid ball of radius  $R$  (or equivalently to the sphere  $\mathcal{S}_R = \partial D_R$ ), tends to the ADM energy in the  $R \rightarrow \infty$  limit. To see this, it is enough to show that  $E_{D_R}$  tends to Komar's expression because it is known that the Komar expression built from the static Killing field  $K^a$  (normalized to one at infinity) tends to the ADM energy (see e.g. [21]). Recall that Komar's integral on a closed spacelike 2-surface  $\mathcal{S}$  (with a timelike and spacelike unit normal,  $t^a$  and  $v^a$ , respectively, and satisfying  $t_a v^a = 0$ ) has the form

$$\mathcal{I}_{\mathcal{S}}[K^e] := \frac{1}{\kappa} \oint_{\mathcal{S}} \nabla^{[a} K^{b]} \frac{1}{2} \varepsilon_{abcd} = \frac{2}{\kappa} \oint_{\mathcal{S}} (v^a t^b \nabla_{[a} K_{b]}) d\mathcal{S}, \quad (3.6)$$

where  $d\mathcal{S} := \frac{1}{2} t^e v^f \varepsilon_{efcd}$  is the induced area element on  $\mathcal{S}$ . Using the Killing equation and the form  $K_a = \exp(\frac{\phi}{c^2}) t_a$  of the Killing field, the integrand can be written as

$$2v^a t^b \nabla_{[a} K_{b]} = 2v^a (\nabla_a K_b) t^b - v^a t^b (\nabla_a K_b + \nabla_b K_a) = 2 \exp\left(\frac{\phi}{c^2}\right) v^a D_a \phi. \quad (3.7)$$

Substituting this into (3.6) and taking into account that  $\exp(\frac{\phi}{c^2}) \rightarrow 1$  at infinity, we find that  $\lim_{R \rightarrow \infty} E_{D_R} = \lim_{R \rightarrow \infty} \mathbb{I}_{\mathcal{S}_R}[K^a]$ .

### 3.3 Static spherically symmetric configurations: Explicit form and positivity

Let the line element of the spatial 3-metric be written as  $dh^2 = -e^{2\alpha}dr^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2)$  for some functions  $R$  and  $\alpha$  of  $r$  and regular at the origin  $r = 0$  with  $R(0) = 0$ . Then the components of the outward pointing unit normal  $v^a$  of the  $\mathcal{S}_r := \{r = \text{const}\}$  2-surfaces are  $v^e = e^{-\alpha}\delta_1^e$ , and the corresponding extrinsic curvature is proportional to the induced 2-metric:  $\nu_{ab} = \frac{R'}{R}e^{-\alpha}(-R^2(\delta_a^2\delta_b^2 + \sin^2\theta\delta_a^3\delta_b^3))$ . Here the prime denotes the derivative with respect to  $r$ . The curvature scalar of the spatial 3-metric is

$${}^3\mathcal{R} = \frac{2}{R^2} \left( 1 + 2RR'e^{-2\alpha}\alpha' - 2RR''e^{-2\alpha} - (R')^2e^{-2\alpha} \right); \quad (3.8)$$

while the curvature scalar of the intrinsic 2-metric on  $\mathcal{S}_r$  is  ${}^2\mathcal{R} = 2/R^2$ . (To avoid confusion, in this subsection the scalar curvatures are denoted by  $\mathcal{R}$ .)

To give an explicit form of the ‘effective energy’ in terms of the geometrical quantities defined on the 2-surface  $\mathcal{S}_r$ , we use not only the trace, but also the  $v^av^b$  component of the evolution equation (3.3). The former is

$$0 = D_a D^a f + f \left( \frac{1}{2} \kappa(\mu + 3p) - \lambda \right) = -e^{-\alpha} (e^{-\alpha} f')' - 2 \frac{R'}{R} e^{-2\alpha} f' + f \left( \frac{1}{2} \kappa(\mu + 3p) - \lambda \right),$$

while the latter is

$$\begin{aligned} 0 &= v^a v^b D_a D_b f + f \left( {}^3G_{ab} v^a v^b + \kappa \sigma_{ab} v^a v^b - \frac{1}{2} \kappa(\mu + 3p) \right) = \\ &= e^{-\alpha} (e^{-\alpha} f')' + f \left( \frac{1}{R^2} - \left( \frac{R'}{R} \right)^2 e^{-2\alpha} + \kappa T_{ab} v^a v^b - \frac{1}{2} \kappa(\mu + 3p) \right). \end{aligned}$$

Here, in the derivation of the second equation, we used the constraint (3.4) and the expression  ${}^3G_{ab} v^a v^b = \frac{1}{2}({}^2\mathcal{R} + \nu_{ab}\nu^{ab} - \nu^2)$  for the  $v^a v^b$  component of the Einstein tensor of the 3-space in terms of the intrinsic and the extrinsic curvatures of  $\mathcal{S}_r$ . Comparing these two and assuming that  $R' \neq 0$ , we obtain the expression of  $f^{-1}f'$  in terms of  $R$ ,  $\alpha$  and  $T_{ab}v^a v^b$ . However, this is essentially the integrand of the 2-surface integral in (3.5),  $v^e D_e \phi = c^2 e^{-\alpha} f^{-1} f'$ , yielding the explicit form of  $E_r$  in spherically symmetric spacetimes:

$$E_r = \frac{4\pi}{\kappa} \frac{R}{R'} e^{\alpha} \left( R^2 (\kappa T_{ab} v^a v^b - \lambda) + 1 - (R')^2 e^{-2\alpha} \right). \quad (3.9)$$

This formula can be simplified slightly if we choose  $r$  to be the areal coordinate, i.e.  $r = R$ , and hence  $R' = 1$ . (3.9) gives  $E_r$  *algebraically* in terms of the functions in the 3-metric and the radial pressure  $T_{ab}v^a v^b$  on the surface  $\mathcal{S}_r$  of radius  $r$ .

In the rest of this subsection we prove that this energy expression is non-negative if the matter fields satisfy the ‘energy conditions’  $(\kappa T_{ab} + \lambda g_{ab})t^a t^b \geq 0$  and  $(\kappa T_{ab} + \lambda g_{ab})v^a v^b \geq 0$ . Moreover, we show that, under the slightly stronger conditions  $(\kappa T_{ab} + \lambda g_{ab})t^a t^b \geq (\kappa T_{ab} + \lambda g_{ab})v^a v^b \geq 0$ , the vanishing of  $E_r$  implies  $\kappa T_{ab} = -\lambda g_{ab}$  and the flatness of the Cauchy development of the ball with radius  $r$ . Clearly,  $(\kappa T_{ab} + \lambda g_{ab})v^a v^b \geq 0$  ensures the



non-negativity of the first term between the brackets in (3.9). To show that the second term dominates the third we use the constraint equation (3.4). Substituting (3.8) into (3.4) and multiplying by  $R^2 R'$ , we obtain  $(R(R')^2 e^{-2\alpha})' = R' - R^2 R'(\kappa\mu + \lambda)$ . Integrating this from zero to  $r$  and using  $(\kappa T_{ab} + \lambda g_{ab})t^a t^b \geq 0$ , we obtain

$$R(R')^2 e^{-2\alpha} = R - \int_0^r (\kappa\mu + \lambda) R^2 R' dr \leq R; \quad (3.10)$$

i.e. that  $(R')^2 \leq e^{2\alpha}$ , and hence that  $E_r \geq 0$ . (Here we used our previous assumption that  $R$  is strictly monotonically increasing, i.e. no minimal or maximal 2-surface is present.)

Conversely, by the pointwise non-negativity of  $\kappa T_{ab} v^a v^b - \lambda$  and of  $e^\alpha - (R')^2 e^{-\alpha}$  from  $E_r = 0$  it follows their vanishing *on the 2-surface  $\mathcal{S}_r$* , i.e. in particular that  $R' = e^\alpha$ . Substituting this to (3.10) we obtain that  $\kappa\mu + \lambda = 0$  *on the whole 3-ball of radius  $r$* , and hence that  $R' = e^\alpha$  also on the whole 3-ball (and not only on its boundary). Substituting this into the line element it becomes flat, i.e. the initial data set on the ball of radius  $r$  is the trivial one, and hence its Cauchy development in the spacetime is also flat. Finally, by the stronger energy condition,  $\kappa T_{ab} t^a t^b = -\lambda$  implies  $\kappa T_{ab} v^a v^b = \lambda$ , i.e.  $\kappa T_{ab} = -\lambda g_{ab}$ .

### 3.4 Comparison with other round sphere expressions

Since the Misner–Sharp energy appears as a mass expression in the study of equilibrium states of cold, spherically symmetric stars (see e.g. Appendix 1 of [19]), this became the more or less generally accepted definition of quasi-local energy on round spheres (i.e. on spherically symmetric 2-surfaces in spherically symmetric spacetimes). (Note that by spherical symmetry the spatial part of the energy-momentum is expected to be zero, and hence the mass is equal to energy.) In the line element of the previous subsection this takes the form  $E_{MS}(r) = \frac{4\pi}{\kappa} r(1 - e^{-2\alpha})$ , where  $r$  is the *areal* coordinate; while the Brown–York energy is  $E_{BY}(r) = \frac{8\pi}{\kappa} r(1 - e^{-\alpha})$ . (N.B.: On round spheres the Bartnik, the Dougan–Mason and the Penrose masses and the Hawking, the Geroch and the Kijowski energies reduce to  $E_{MS}(r)$ . On the other hand, the Brown–York energies (with all three choices for the reference configurations), the Epp, the Kijowski–Liu–Yau (which is Kijowski’s free energy) and the Wang–Yau expressions reduce to  $E_{BY}(r)$ . For the details see e.g. [17].) Since the Hawking energy is a gauge invariant measure of the bending of the light rays orthogonal to the 2-surface (see subsection 6.1.1 of [17]), the Misner–Sharp energy can also be interpreted in this way.

On the other hand, the ‘effective’ quasi-local energy was introduced as the integral of the effective source for the Newtonian potential *seen by the static observers*, i.e. there is a *different concept of energy* behind this: it is a measure of the effective source of gravity, including also the gravitational self-interaction. This yields that, in addition to the extra pressure and cosmological terms in (3.9), it has an extra overall weight function  $e^\alpha$  with respect to  $E_{MS}(r)$ . Since by the positivity proof above it is not less than 1, the energy  $E_r$  is never less than the Misner–Sharp energy.

In the Schwarzschild solution the Misner–Sharp energy is the constant  $\frac{8\pi}{\kappa} \mathfrak{m}$  for any  $r > 2\mathfrak{m}$ , the ‘effective’ quasi-local energy is  $E_r = \frac{8\pi}{\kappa} \frac{\mathfrak{m}}{\sqrt{1 - \frac{2\mathfrak{m}}{r}}}$ , while the Brown–York energy is  $E_{BY}(r) = \frac{8\pi}{\kappa} r(1 - \sqrt{1 - \frac{2\mathfrak{m}}{r}})$ . Thus, the Schwarzschild mass parameter  $\mathfrak{m}$  is only the ‘bare’ mass, while in  $E_r$  this ‘bare’ mass is ‘dressed’ by the inverse local redshift factor and it tends to the ADM energy from above. Similarly,  $E_{BY}(r)$  is also monotonically

decreasing. To see their more detailed asymptotic structure let us expand them as a power series of  $1/r$  near infinity. We find

$$E_{BY}(r) = \frac{8\pi}{\kappa} \mathfrak{m} \left( 1 + \frac{1}{2} \frac{\mathfrak{m}}{r} \right) + O(r^{-2}), \quad (3.11)$$

$$E_r = \frac{8\pi}{\kappa} \mathfrak{m} \left( 1 + \frac{\mathfrak{m}}{r} \right) + O(r^{-2}); \quad (3.12)$$

which, by  $\mathfrak{m} \sim Gc^{-2}$  (see the last sentence of subsection 2.2), are the post-Newtonian expansions at the same time. Thus the Brown–York energy reproduces the Newtonian energy expression even in the  $c^{-2}$  order. On the other hand, by general relativity it is *twice* the Newtonian gravitational energy (more precisely the sum of the Newtonian gravitational energy and the average pressure) that appears as the source of gravity in the first post-Newtonian order (see (3.1) and the subsequent discussion). The factor 2 in front of the post-Newtonian energy term in (3.12) is simply a manifestation of the contribution of the gravitational spatial stress. Since, however,  $E_r$  is a measure of the source *seen by the static observers*, it diverges at the horizon as it could be expected. Thus our ‘effective’ quasi-local energy may provide a physically reasonable notion of energy in the region where the Killing vector is timelike, i.e. outside the event horizon.

## 4 Discussion

In the literature there exist lists of *a priori* expectations on how a physically reasonable quasi-local mass or energy-momentum expression should behave [17, 22], e.g. at spatial infinity, in the presence of spherical symmetry, or on the event horizon of black holes.

One such additional natural expectation could be the compatibility with the results in the relativistically corrected Newtonian theory, where the quasi-locally defined energy of the matter plus gravity system tends to the total energy, measured at infinity, as a *strictly decreasing* set function. Thus, to be able to make this comparison, we need to define the Newtonian limit of general relativity. One of the several possible definitions is based on static spacetimes that are asymptotically flat at spatial infinity [19]. Then we can expand every quantity and formula as a series of  $c^{-2}$ , and while the zeroth order terms give the Newtonian approximation, the coefficients of  $c^{-2k}$  are the  $k$ th order post-Newtonian corrections. In fact, since near spatial infinity in an asymptotically flat spacetime the matter fields and the dynamics of both the matter and the geometry die off rapidly, the quasi-local energy-momentum/mass expressions could be expected to behave like in *static* asymptotically flat spacetimes. Though this notion of post-Newtonian approximation is more restrictive than the usual one (see [23]), the advantage of this is that the presence of the *static* Killing field provides extra geometric structures that make the subsequent analysis technically much easier and unique. In particular, they make it possible to define *energy* at the quasi-local level and to be able to compare energies (and not only masses) associated with *different* 2-surfaces unambiguously.

Since in static spacetimes the gravitational contribution to the total energy is *negative definite*, the quasi-local expressions should tend to the ADM expression as *strictly decreasing* functions. Therefore, in particular, if  $E(r)$  is any such gravitational energy expression evaluated on a large sphere of radius  $r$  near spatial infinity and  $E_r$  denotes the ‘effective’ energy, also at  $r$ , then the asymptotic form of the former could be expected

e.g. to be  $E(r) = E_r + O(r^{-2})$  or  $E(r) = E_{BY}(r) + O(r^{-2})$ , depending on the concept of energy that is behind the actual notion  $E(r)$  of quasi-local energy.

However, this new requirement is in conflict with two of the previous ones in [17, 22]. First, this contradicts to the expectation that for round spheres the quasi-local energy should reduce to the Misner-Sharp energy since the latter is *increasing* (or constant). Thus in static spherically symmetric configurations  $E_r$  could be an alternative to the Misner-Sharp expression.

Second, if the quasi-local mass should really tend to the ADM mass as a *strictly decreasing* set function near spatial infinity, then the Schwarzschild example shows that the quasi-local mass at the event horizon *cannot* be expected to be the irreducible mass. In fact, since both the ADM and irreducible masses are  $\frac{8\pi}{\kappa}m$  and the quasi-local mass must be strictly decreasing, there would have to be a closed 2-surface between the horizon and the spatial infinity on which the quasi-local mass would take its maximal value. However, it does not seem why such a (geometrically, and hence, physically) distinguished 2-surface should exist.

This work was partially supported by the Hungarian Scientific Research Fund (OTKA) grant K67790 and by the Royal Society of New Zealand with Marsden grant UOO-09-022.

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